# Conformal Mesh Parameterization

Richard Liu

May 17th, 2022

# Introduction

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**Historical motivation:** cartography

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**Historical motivation:** cartography

Turns out it is **impossible** to make a 2D map of the Earth without some distortion and/or cutting.

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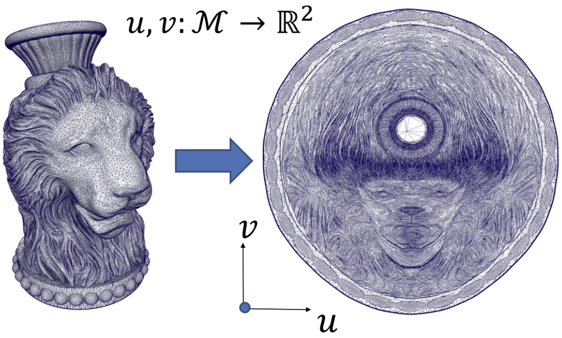

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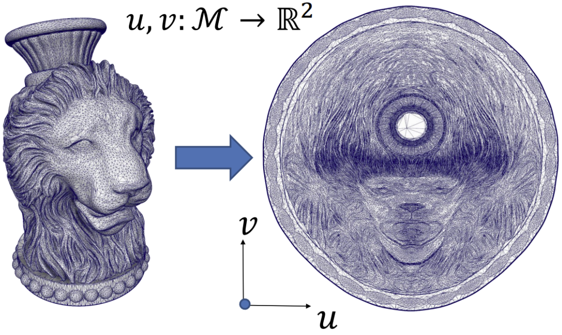



**Conformal mesh parameterization:** "angle-preserving" maps

# Mesh Parameterization Applications

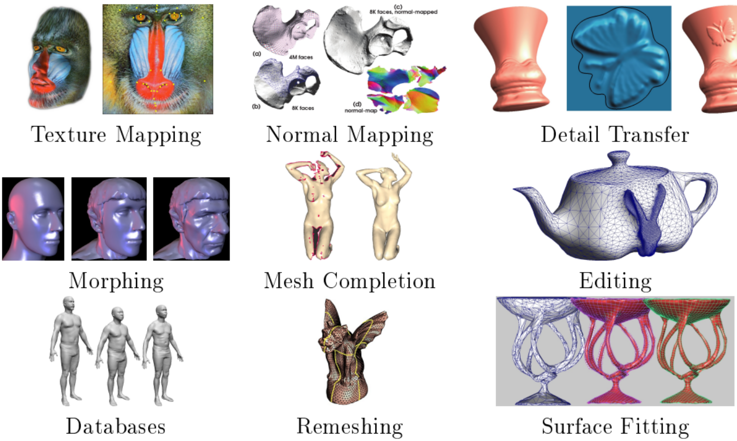
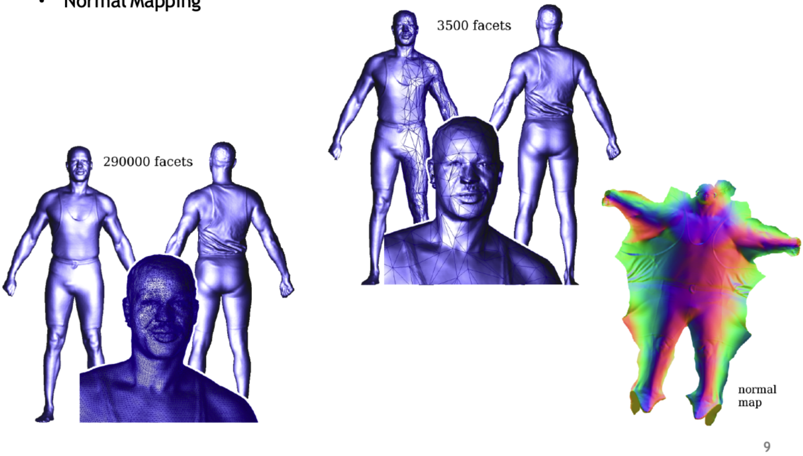



Figure: Parameterization Applications

# Mesh Parameterization Applications

- Normal Mapping



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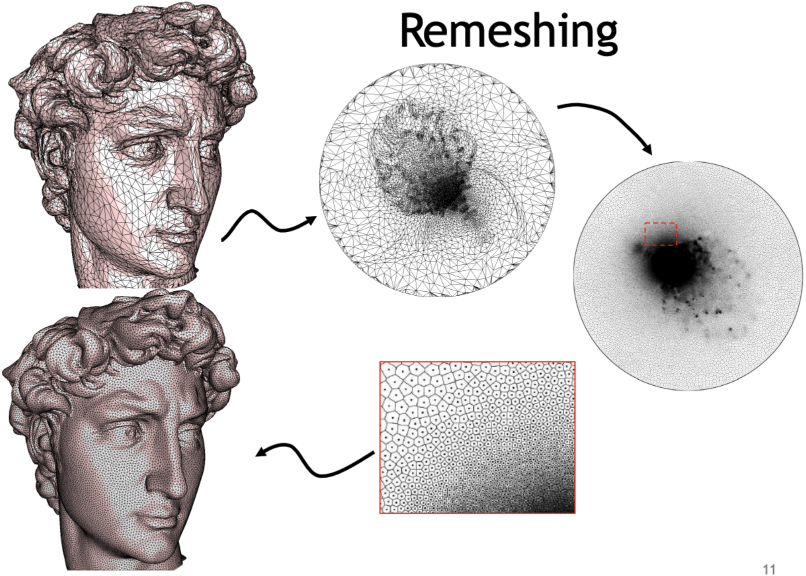

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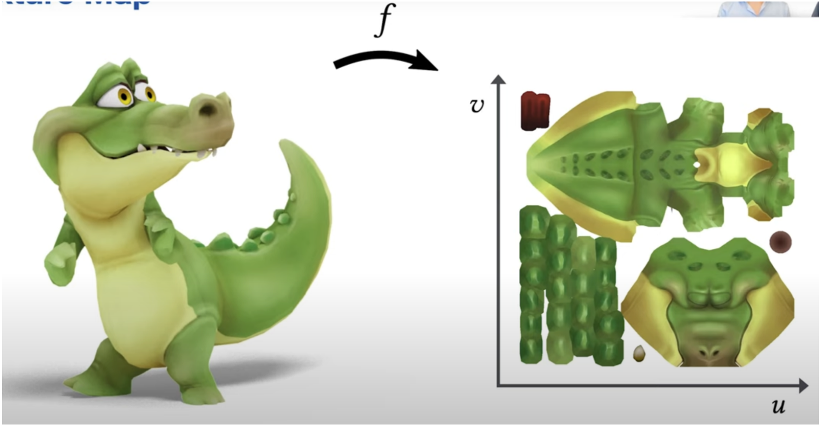


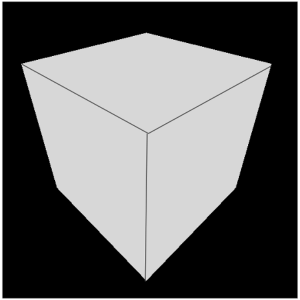

Figure: Texture Mapping

## Texture Mapping

Image from Vallet and Levy, techreport INRIA

# Introduction

**Recall:** in assignment 2, you were asked to generate a texture map for each face of the cube



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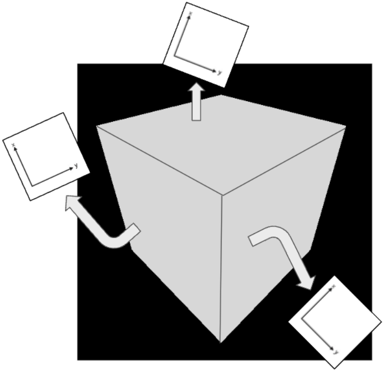

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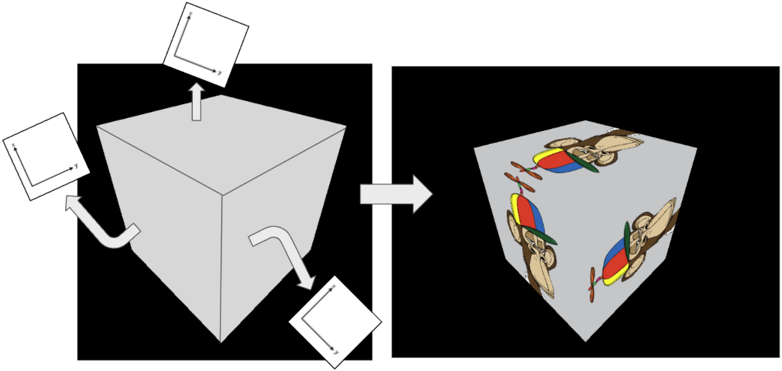

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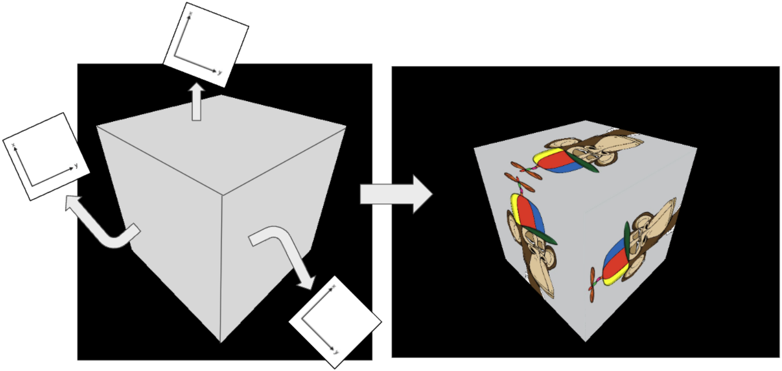
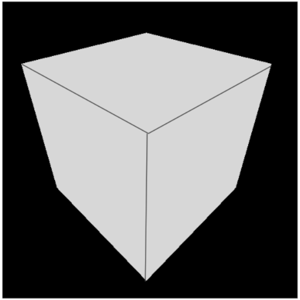


**This is an example of a parameterization. Namely, a piecewise linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$.**

# Introduction

Okay, but what if we want to paste the **whole image** onto the whole cube?



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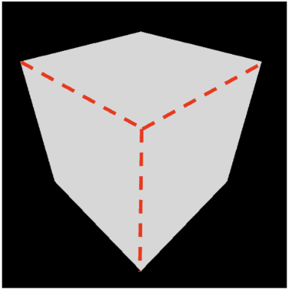

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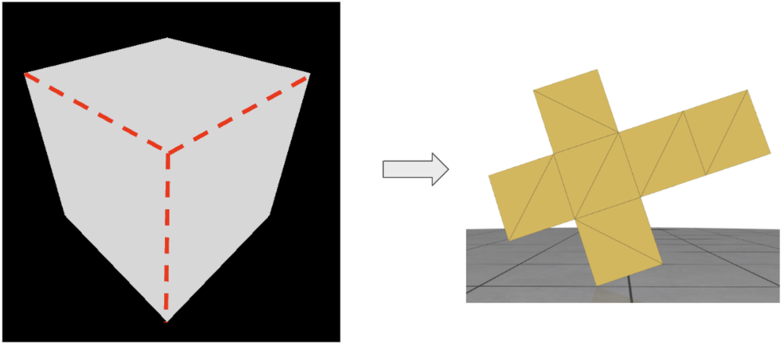

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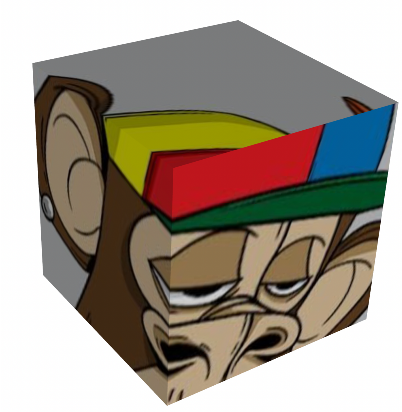

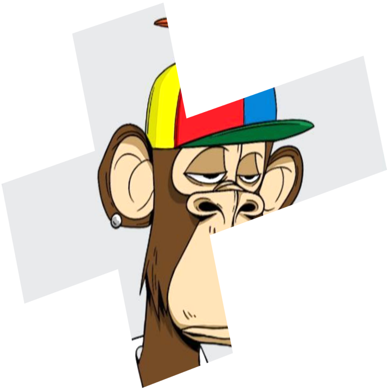

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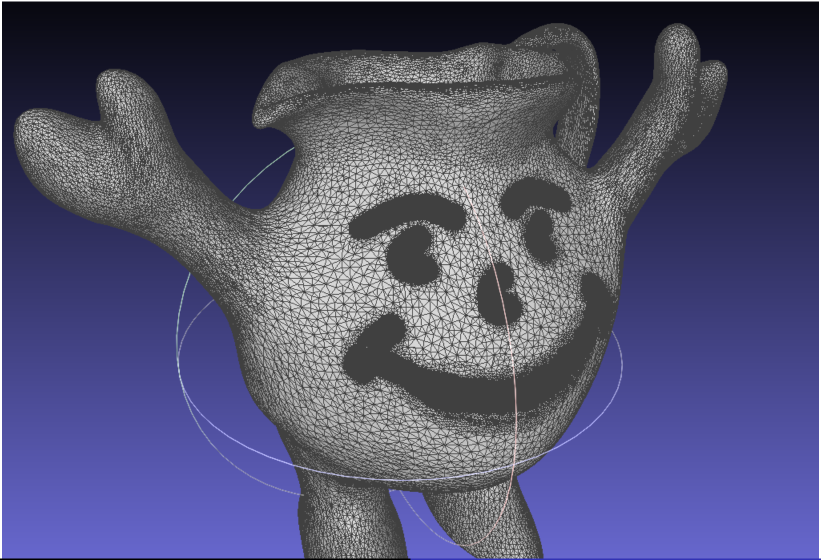
Okay, but what about something a bit more challenging?



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Let's do some math.

# Mathematical Framework

Given a function $f : X \to Y$,

**Definition**

f is **injective** or **one-to-one**, if $\forall x, x' \in X, f(x) = f(x') \Rightarrow x = x'$.

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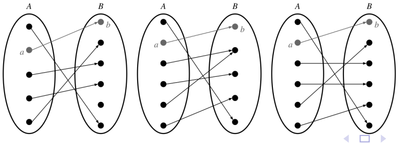


(a) ***Injective*** but **non-surjective** (b) **Non-Injective** but ***surjective*** (c) **Injective** and **surjective** (***bijective***)

# Mathematical Framework

## Definition

Let $\Omega \subset \mathbb{R}^2$ be a **simply connected** (without holes) region. Let $f : \Omega \to \mathbb{R}^3$ be continuous and injective. The image of $f$ is called a **surface**

$$S = f(\Omega) = \{f(u, v) : (u, v) \in \Omega\}$$

We say that $f$ is a **parameterization** of $S$ over the **parameter domain** $\Omega$.

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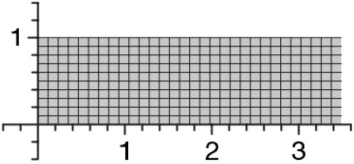
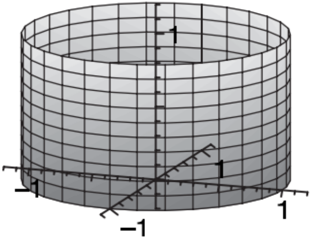
**Note 2:** In practice we care about the inverse map $f^{-1} : S \to \Omega$, but we formulate in this way for the math to make sense.

# Mathematical Framework

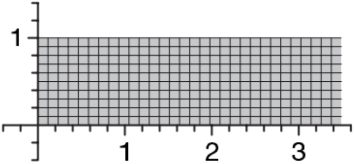
## Example

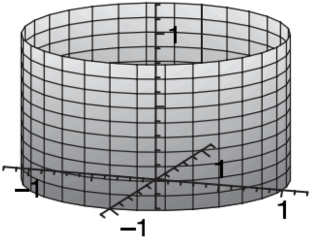



- Parameter domain: $\Omega = \{(u, v) \in \mathbb{R}^2 : u \in [0, 2\pi), v \in [0, 1]\}$

**Example**

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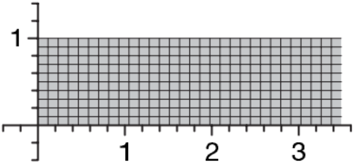


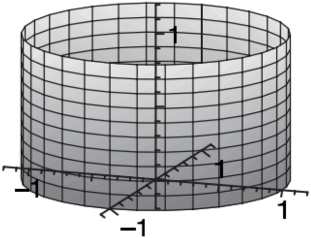


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- Parameterization: $f(u, v) = (\cos u, \sin u, v)$
- Inverse: $f^{-1}(x, y, z) = (\arccos x, z)$

# Mathematical Framework

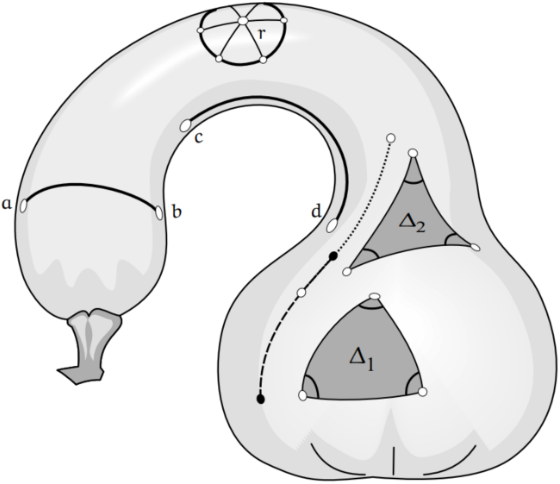
## Remark

A parameterization $f : \Omega \to S$ is never unique. Given any bijection $\gamma : \Omega \to \Omega$, $g = f \circ \gamma$ is a parameterization of $S$ over $\Omega$.

# Mathematical Framework

We can use $f$ for deriving some key **intrinsic surface properties**, or properties that are independent of how the surface sits in space (extrinsic geometry).

# Mathematical Framework




**[1.9]** *The* **intrinsic geometry** *of the surface of a crookneck squash:* **geodesics** *are the equivalents of straight lines, and triangles formed out of them may possess an angular excess of either sign, depending on how the surface bends:* $\mathcal{E}(\Delta_1) > 0$ *and* $\mathcal{E}(\Delta_2) < 0$.

# Mathematical Framework

**Definition**

A parameterization $f : \Omega \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ is **regular** if the tangent vectors $f_u = \frac{\partial f}{\partial u}$ and $f_v = \frac{\partial f}{\partial v}$ are always linearly independent.

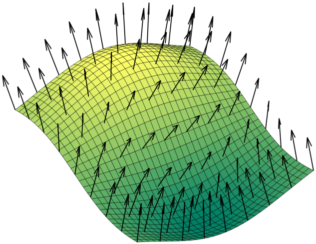
**Note:** $f_u$, $f_v$ are functions from $\mathbb{R}^2$ to $\mathbb{R}^3$ and span the local tangent plane.

# Mathematical Framework

**Definition**

Given a regular parameterization $f$, the **surface normal** $n_f$ is defined as

$$n_f = \frac{f_u \times f_v}{||f_u \times f_v||}$$



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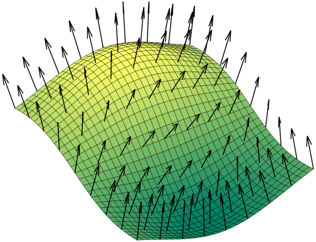
**Note:** regularity is required for $n_f$ to be nonzero everywhere.



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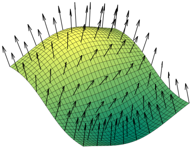

The surface normal is an **intrinsic property**, which means it is **the same independent of the parameterization f.**

# Mathematical Framework

We can also apply $f$ towards deriving the **first and second fundamental forms**, $I_f$ and $II_f$.

# Mathematical Framework

We can also apply $f$ towards deriving the **first and second fundamental forms**, $I_f$ and $II_f$.

They are fundamental precisely because they characterize the key metric properties of a surface, such as the **gaussian curvature**, **mean curvature**, and **surface area**. They are also essential for telling us about how the tangent vectors stretch, i.e. **parameterization distortion.**

# Mathematical Framework

## Definition

Given parameterization $f$, the **first fundamental form** is defined as

$$\mathrm{I}_f = \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

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## Area of a Surface

Given parameterization $f : \Omega \to S$, the area $A(S)$ can be found

$$A(S) = \int_\Omega \sqrt{\det(\mathrm{I}_f)} du dv$$

# Mathematical Framework

**Definition**

Given a twice-differentiable parameterization $f$, the **second fundamental form** is defined as

$$\mathrm{II}_f = \begin{pmatrix} f_{uu} \cdot n_f & f_{uv} \cdot n_f \\ f_{uv} \cdot n_f & f_{vv} \cdot n_f \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

# Mathematical Framework

**Definition**

The Gaussian curvature K is

$$K = \det(\mathrm{I}_f^{-1}\mathrm{II}_f) = \frac{\det \mathrm{II}_f}{\det \mathrm{I}_f} = \frac{LN - M^2}{EG - F^2} = \kappa_1\kappa_2$$

where $\kappa_1$, $\kappa_2$ are the **principal curvatures** (the curvatures along the directions the surface bends the most).

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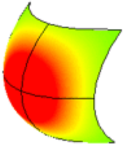
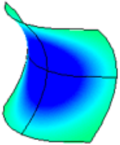
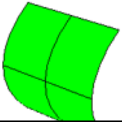
**Positive curvature**
A positive Gaussian curvature value means the surface is bowl-like.



**Negative curvature**
A negative value means the surface is saddle-like.



**Zero curvature**
A zero value means the surface is flat in at least one direction. (Planes, cylinders, and cones have zero Gaussian curvature).



# Mathematical Framework

**Definition**

The mean curvature S is

$$S = \frac{1}{2}\text{trace}(\mathrm{I}_f^{-1}\mathrm{II}_f) = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{\kappa_1 + \kappa_2}{2}$$

where $\kappa_1$, $\kappa_2$ are the **principal curvatures** (the curvatures along the directions the surface bends the most).

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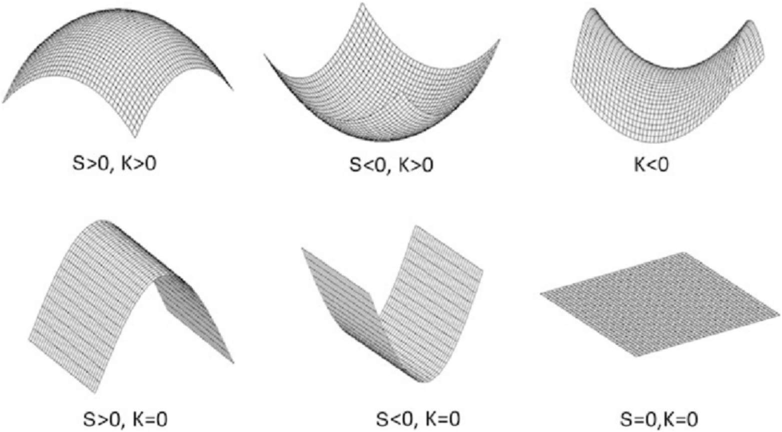
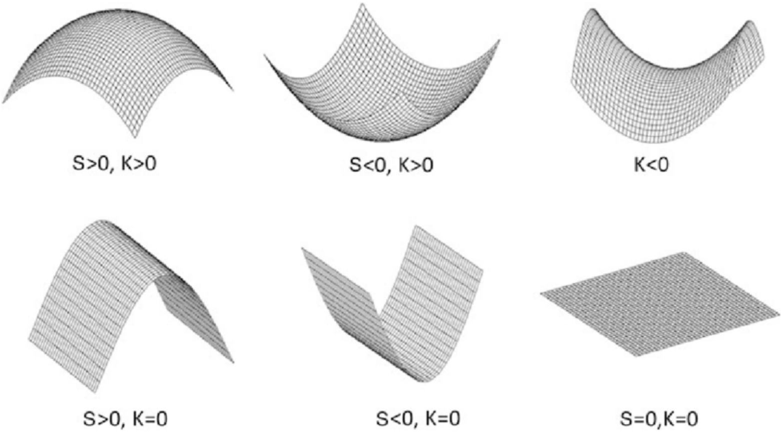



K = Gaussian curvature (**intrinsic**)
S = Mean curvature (**extrinsic**)

# Mathematical Framework




K = Gaussian curvature (**intrinsic**)
S = Mean curvature (**extrinsic**) $\rightarrow$ Careful: different notion of instrinsic. invariant to parameterization but **not** to isometry

# Mathematical Framework

**Definition**

A surface S is **developable** if $\forall p \in S$, $K(p) = 0$, i.e. the Gaussian curvature is 0 everywhere on $S$.

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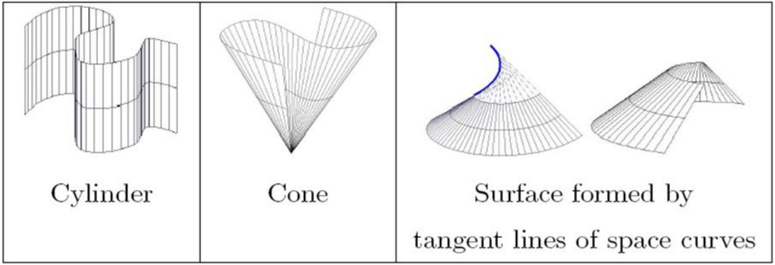
**Theorem**

*(Gauss, 1827) Globally isometric parameterizations (from 3D to 2D) only exist for* developable *surfaces (i.e. $K = 0$ everywhere)*

# Developable Surface

Three types of developable surfaces



# Mathematical Framework

**Definition**

The **Jacobian** of parameterization $f$ is the 3 x 2 matrix of partial derivatives of $f$.

$$J_f = (f_u, f_v)$$

# Mathematical Framework

**Definition**

For any $m \times n$ matrix $J$, the **singular value decomposition** (SVD) is given by

$$J = U \Sigma V^T$$

where $\Sigma$ is an $m \times n$ diagonal matrix, and $U$ and $V$ are $m \times m$ and $n \times n$ orthonormal matrices, respectively.

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By the above, the SVD of the Jacobian is

$$J_f = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

where $\sigma_1$, $\sigma_2$ are the **singular values**.

# Mathematical Framework

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**Remark**

We can write the first fundamental form as

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It is clear $\mathrm{I}_f$ is **symmetric**.

Thus the eigenvalues of $\mathrm{I}_f$ are given by

$$\lambda_{1,2} = \frac{1}{2}((E+G) \pm \sqrt{4F^2 + (E-G)^2}$$

.

**Remark**

For a matrix $A$, the singular values are the square roots of the eigenvalues of $A^T A$.

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The singular values of $J$ can be found using the eigenvalues of $\mathrm{I}_f$

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# Mathematical Framework

**Remark**

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The singular values of $J$ can be found using the eigenvalues of $\mathrm{I}_f$

$$\sigma_1 = \sqrt{\lambda_1}$$
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$\sigma_1$ and $\sigma_2$ tell us **everything** about the **metric distortion** induced by the parameterization.

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Parameterizations induce distortion in **lengths**, which can be further divided into distortion in **angles** and distortion in **areas**.

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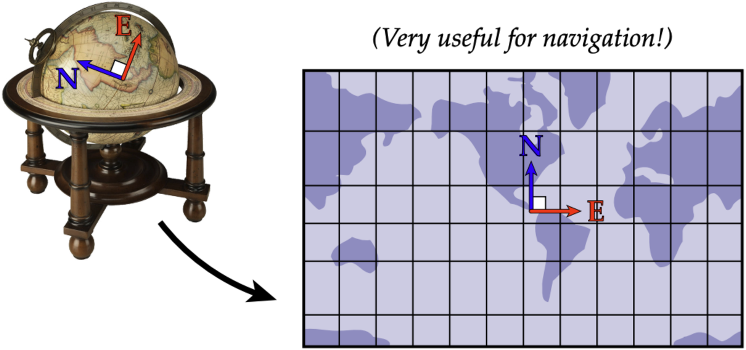
- Amazing fact: can always make a map that exactly preserves **angles**.



# Properties of Parameterizations

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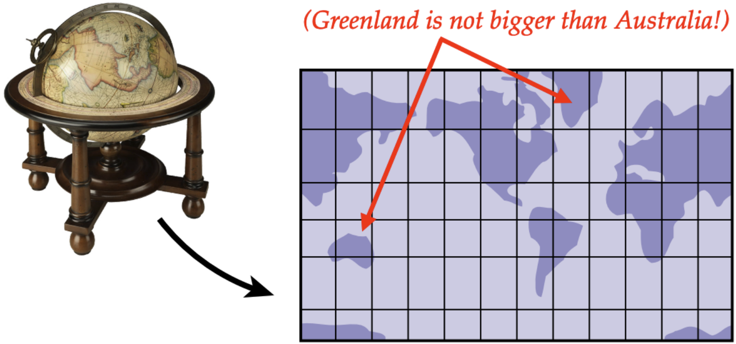
- However, **areas** may be badly distorted…



# Properties of Parameterizations

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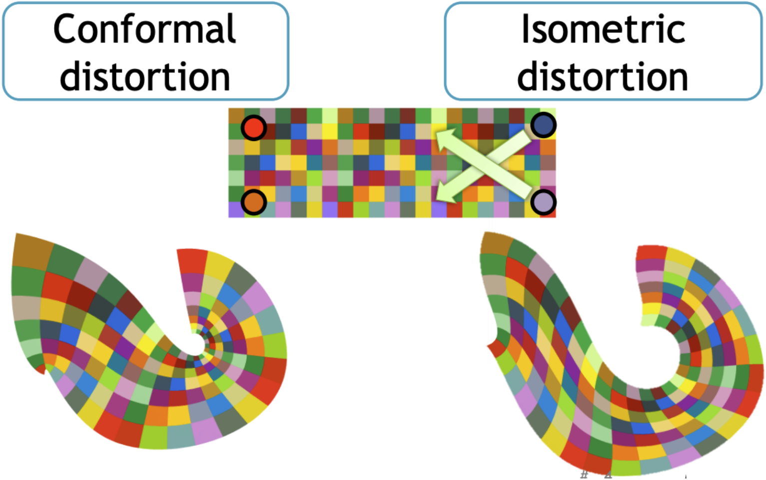

Some intuition: the singular values induce **scaling** of the tangent vectors of the mapped surface.

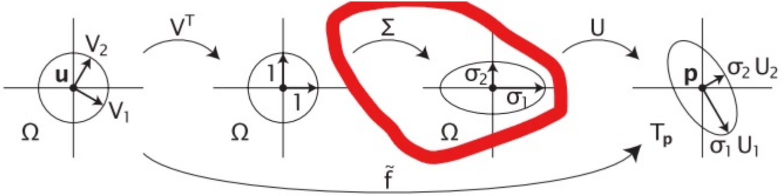


Figure: SVD Decomposition of mapping $\tilde{f}$

# Properties of Parameterizations

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**Nope.** Recall:

**Theorem**

*(Gauss, 1827) Globally isometric parameterizations (from 3D to 2D) only exist for* developable *surfaces (i.e. $K = 0$ everywhere)*

# Properties of Parameterizations

So how to find the "best" parameterization?

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Take bivariate non-negative function $E : \mathbb{R}_+^2 \to \mathbb{R}_+$ that takes local distortion measures $\sigma_1$ and $\sigma_2$, and has minimum defined according to objective.

$$E(f) = \int_\Omega E(\sigma_1(u, v), \sigma_2(u, v)) dudv / A(\Omega)$$

# Discrete Setting

Note that everything up until this point has been formulated in the continuous setting. So what changes when we consider the discrete mesh setting?

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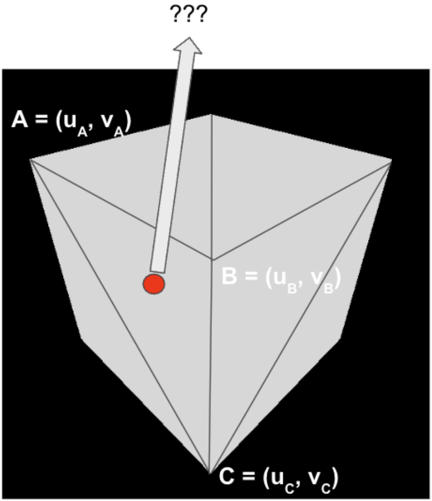
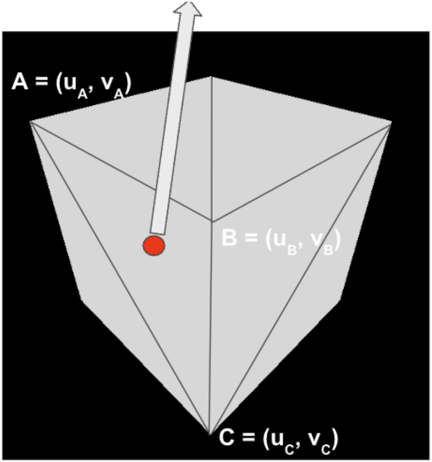
$f$ can now be considered a **piecewise linear map**. Specifically, we only have to care about how **vertices are mapped to the plane**.

# Discrete Setting



# Discrete Setting

$$p_{UV} = \alpha A + \beta B + \gamma C$$



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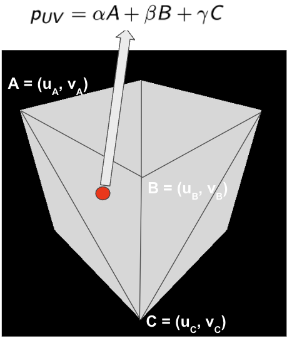


Our parameterization function is now $f : \Omega \to V$ where $V$ is the set of vertex positions in the mesh.

# Mesh Parameterization Properties

We already mentioned **conformal**, **equiareal**, and **isometric** maps. Another important property for applications to meshes is **bijectivity**.

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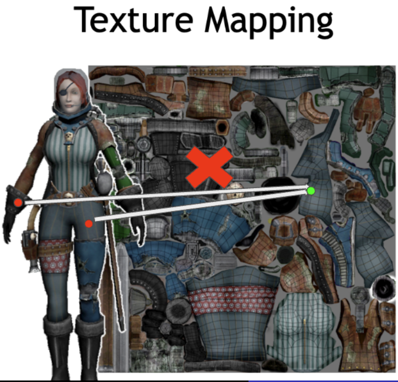
e.g. For texture mapping, want to be able to annotate parts of the texture with reference to unique region of surface



# Mesh Parameterization Properties

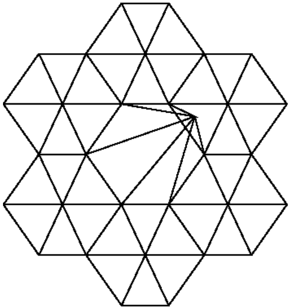
### Definition

A mesh parameterization is **locally injective** if no triangles change orientation ("flip" or "fold over") during the parameterization.

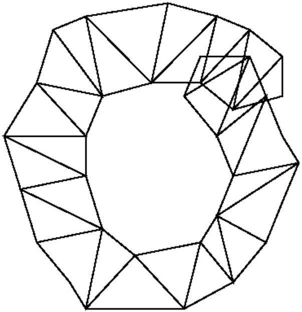
### Definition

A mesh parameterization is **globally bijective** if it is locally injective and the boundary of the parameterization does not intersect itself.

# Mesh Parameterization Properties



Triangle Flip



Boundary Intersection

# LSCM

**Least Squares Conformal Maps for Automatic Texture Atlas Generation**

Bruno Lévy    Sylvain Petitjean    Nicolas Ray    Jérome Maillot*

ISA (Inria Lorraine and CNRS), France

# LSCM

**LSCM.** (Levy et al. 2002) The least squares conformal maps method seeks to minimize the following **conformal energy**

$$E_{LSCM} = E_C = \frac{1}{2} \int_S ||f_v - \text{rot}_{90}(f_u X)||^2 dp = \frac{(\sigma_1 - \sigma_2)^2}{2}$$

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Intuition: the gradient vectors $f_u$ and $f_v$ are **orthogonal** and **have the same norm** ($\sigma_1 = \sigma_2$).

# LSCM

Recall how the singular values of the Jacobian scales tangent vectors.

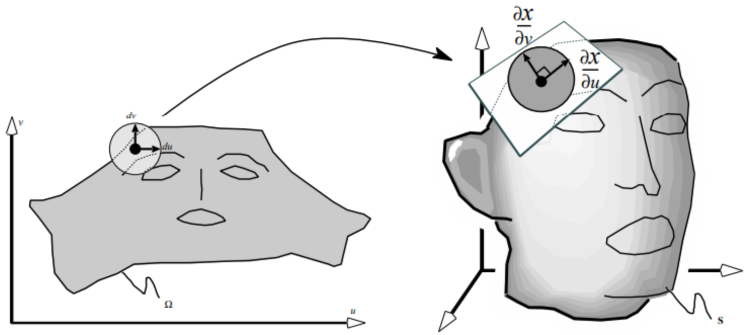


Figure 4.9: A conformal parameterization transforms an elementary circle into an elementary circle.

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**Paper's perspective:** Cauchy-Riemann equations from complex analysis.

**Theorem**

*If a function $f(z) = u(x, y) + iv(x, y)$ is complex differentiable (holomorphic), then it satisfies the partial differential equations*

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**Holomorphic implies conformality. This is equivalent to a condition on the Jacobian.**

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We want the Jacobian

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

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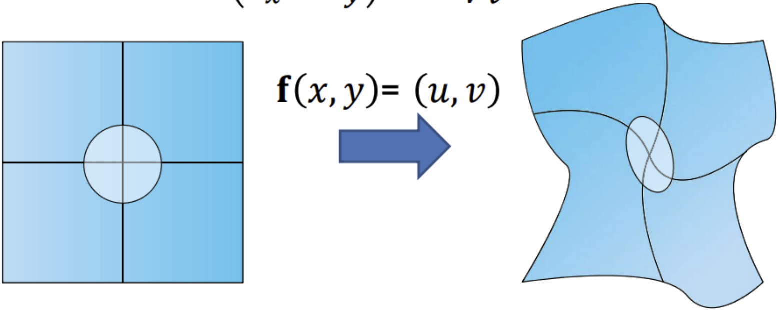

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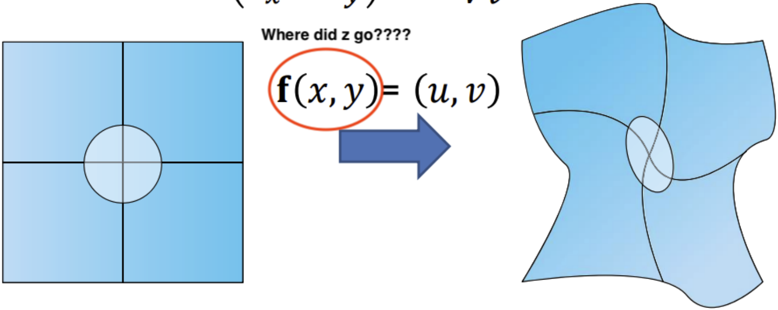

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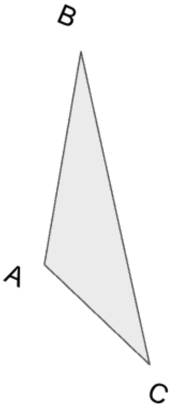

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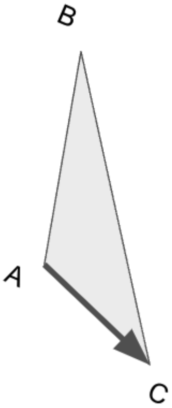

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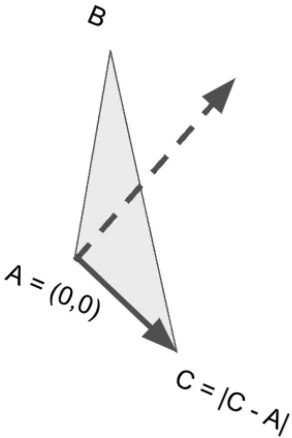

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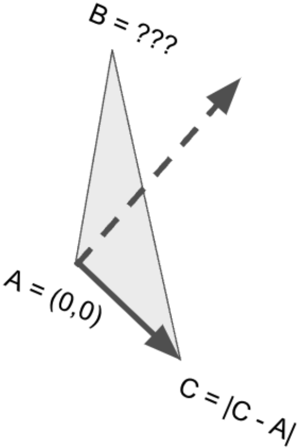

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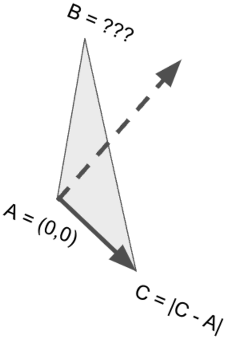


**Use vector projection to get magnitude of (B-A) in the direction of the local coordinate vectors.**

# LSCM

One last problem: the current distortion formula has a degenerate solution.

**LSCM - Least Squares Conformal Map**

**We want the Jacobian** **to be a similarity matrix**

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \qquad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

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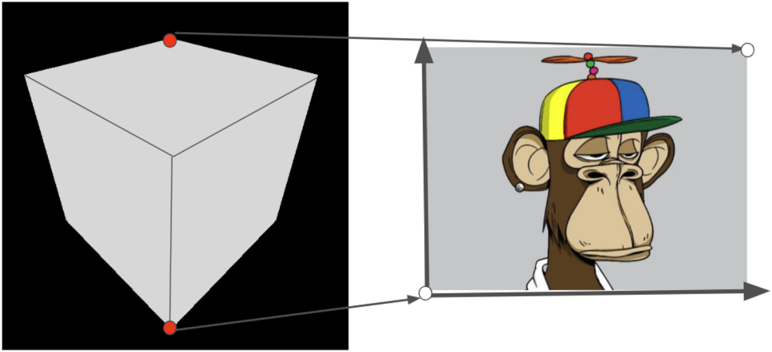


# LSCM

Finally, we can write the LSCM objective function as a linear least squares problem.

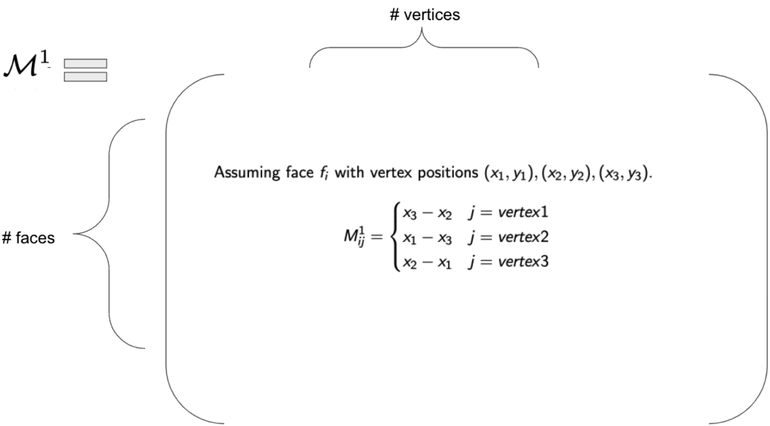
Rewriting the objective function with only real matrices and vectors yields

$$C(\mathbf{x}) = \|\mathcal{A}\mathbf{x} - \mathbf{b}\|^2 , \tag{4}$$

with

$$\mathcal{A} = \begin{pmatrix} \mathcal{M}_f^1 & -\mathcal{M}_f^2 \\ \mathcal{M}_f^2 & \mathcal{M}_f^1 \end{pmatrix}, \quad \mathbf{b} = -\begin{pmatrix} \mathcal{M}_p^1 & -\mathcal{M}_p^2 \\ \mathcal{M}_p^2 & \mathcal{M}_p^1 \end{pmatrix} \begin{pmatrix} \mathbf{U}_p^1 \\ \mathbf{U}_p^2 \end{pmatrix},$$

# LSCM




Assuming face $f_i$ with vertex positions $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

$$M^1_{ij} = \begin{cases} x_3 - x_2 & j = vertex1 \\ x_1 - x_3 & j = vertex2 \\ x_2 - x_1 & j = vertex3 \end{cases}$$

# LSCM

$\mathcal{M}^2 =$

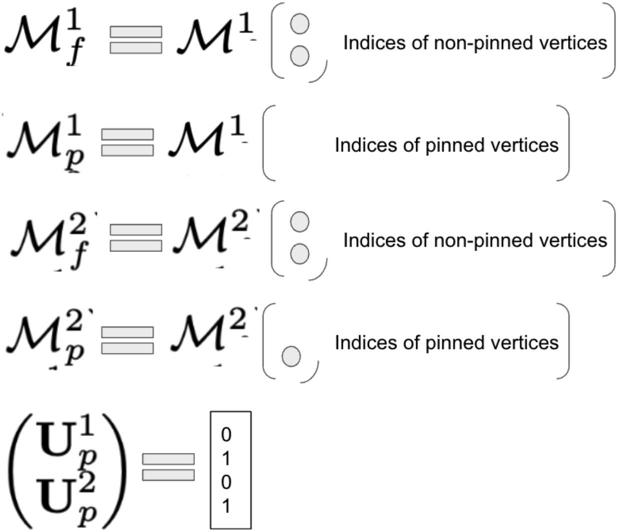
# vertices

# faces

Assuming face $f_i$ with vertex positions $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

$$M_{ij}^2 = \begin{cases} y_3 - y_2 & j = vertex1 \\ y_1 - y_3 & j = vertex2 \\ y_2 - y_1 & j = vertex3 \end{cases}$$

# LSCM


Indices of non-pinned vertices
Indices of pinned vertices
Indices of non-pinned vertices
Indices of pinned vertices
0
1
0
1

# LSCM

LSCM Pseudocode

def LSCM(V: vertices, F: faces):
Choose two vertices (b1, b2) to pin (use vertices that are far apart for better results).
For each triangle, convert the incident vertices to local coordinates (should shared vertices across different triangles).
Construct the **A** and **b** matrices from (4).
Solve the least squares equation for **x** in (4). This will give you back your UV coordinates in a 1D vector in the form ($u_1$, $u_2$, ..., $u_{|}V|$, $v_1$, $v_2$, ..., $v_{|}V|$)

# Thank you!

**Resources**

LSCM

Mesh Parameterization: Theory and Practice (2008)

Mesh Parameterization Methods and Their Applications